is a commutative (not necessarily Cartesian!) diagram, and $\mathscr{F}$ is a sheaf on X. Define the **push-pull map**

$$\beta^{-1}\alpha_*\mathscr{F} \longrightarrow \alpha'_*(\beta')^{-1}\mathscr{F} \tag{2.7.4.2}$$

of sheaves on Y as follows. Start with the identity $(\beta')^{-1}\mathscr{F} \xrightarrow{\sim} (\beta')^{-1}\mathscr{F}$ on W. By adjointness of $((\beta')^{-1}, \beta'_*)$, this is the same as the data of a morphism $\mathscr{F} \to (\beta'_*)(\beta')^{-1}\mathscr{F}$ on X. Apply α_* to get a map $\alpha_*\mathscr{F} \to \alpha_*(\beta'_*)(\beta')^{-1}\mathscr{F}$ on Z. By the commutativity of (2.7.4.1), this is the map $\alpha_*\mathscr{F} \to \beta_*(\alpha'_*)(\beta')^{-1}\mathscr{F}$ on Z. By adjointness of (β^{-1}, β_*), this yields a map (2.7.4.2).

We observe that this entire construction is functorial in $\mathscr{F}$ (i.e., given a map $\mathscr{F} \to \mathscr{G}$ of sheaves on X, we get a certain commutative diagram of sheaves on Y — what is it?). (We will later extend this to $\mathscr{O}$-modules, quasicoherent sheaves, and cohomology, see Exercises 7.2.D(f), 14.6.K, and 18.7.B.)

2.7.F. EXERCISE. We could have defined the push-pull map in a "dual way" starting with the identity $\alpha_*\mathscr{F} \to \alpha_*\mathscr{F}$ on Z, then using adjointness of (α^{-1}, α_*), and continuing from there. Why does this give the *same* definition of the push-pull map?

2.7.5. The support of a sheaf, and the support of a section of a sheaf.

Exercise 2.7.H below gives us an excuse to introduce the notion of *support*, which we use repeatedly later.

2.7.6. *Definition.* Suppose $\mathscr{F}$ is a sheaf (or indeed separated presheaf) of abelian groups on X, and s is a global section of $\mathscr{F}$. Define the **support of the section** s, denoted Supp s, to be the set of points p of X where s has a nonzero germ:

$$\operatorname{Supp} s := \{p \in X \ : \ s_p \neq 0 \text{ in } \mathscr{F}_p\}.$$

We think of this as the subset of X where "the section s lives" — the complement is the locus where s is the 0-section. (Unimportant: We could define this even if $\mathscr{F}$ is a presheaf, but without the inclusion $\mathscr{F}(U) \hookrightarrow \prod_{p \in U} \mathscr{F}_p$ of Exercise 2.4.A, we could have the strange situation where we have a nonzero section that "lives nowhere", because it is 0 "near every point", i.e., is 0 in every stalk.)

2.7.G. EXERCISE (THE SUPPORT OF A SECTION IS CLOSED). Show that Supp s is a closed subset of X.

2.7.7. *Caution: the locus where a continuous function is nonzero is open; the locus where the germ of a function is nonzero is closed.* Basically by the definition of continuity, the locus where the *value* of a continuous function is nonzero is *open*. (More generally, the locus where the value of a function on a locally ringed space is nonzero is open, see Exercise 4.3.F(a).) In contrast, Exercise 2.7.G shows that the locus where the *germ* of a function is nonzero is *closed*. We will try to avoid misunderstanding by using phrases like "f is 0 at p" (the value of f is zero, i.e., $f(p) = 0$) and "f is 0 near p" (the germ of f is zero, i.e., $f = 0$ in $\mathscr{O}_{X,p}$, or equivalently, f is zero in some neighborhood of p).

2.7.8. *Definition.* Define the **support of a sheaf** of groups $\mathscr{G}$ of sets, denoted Supp $\mathscr{G}$, as the locus where the stalks are nontrivial:

$$\operatorname{Supp} \mathscr{G} := \{p \in X \ : \ |\mathscr{G}_p| \neq 1\}.$$

Equivalently, Supp $\mathscr{G}$ is the union of supports of sections over all open sets. Clearly support is a "stalk-local notion", and hence "commutes" with restriction to open sets. (Irrelevant for us: more generally, if the sheaf has value in some category, such as the category of sets, the support can be defined as the points where the stalk is not the terminal object.)

2.7.H. EXERCISE.
(a) Suppose $Z \subset Y$ is a closed subset, and $i \colon Z \hookrightarrow Y$ is the inclusion. If $\mathscr{F}$ is a sheaf of groups on Z, then show that the stalk $(i_*\mathscr{F})_q$ is the one-element group if $q \notin Z$, and $\mathscr{F}_q$ if $q \in Z$.
(b) Suppose Supp $\mathscr{G} \subset Z$ where Z is closed. Show that the natural map $\mathscr{G} \to i_* i^{-1}\mathscr{G}$ is an isomorphism. Thus a sheaf supported on a closed subset can be considered a sheaf on that closed subset.

2.7.9. Extension by zero, an occasional *left adjoint* to the inverse image functor. In addition to always being a left adjoint, π^{-1} can sometimes be a right adjoint, when π is an inclusion of an open subset. We discuss this when we need it, in §23.4.7.