We give the statement of Theorem 23.2.9 for simplicity, as we will only use this version.

23.2.11. Corollary. — *If $\mathscr{A}$ has enough injectives, and F is a left-exact covariant functor $\mathscr{A} \to \mathscr{B}$, then the R^iF (with the δ^i that accompany them) form a universal δ-functor.*

Proof. Each element of $\mathscr{A}$ admits a monomorphism into an injective element; this is just the definition of "enough injectives" (Exercise 23.2.C). Higher derived functors of an injective elements I are always 0: just compute the higher derived functor by taking the injective resolution of I "by itself". □

23.3 Derived functors and spectral sequences

A number of useful facts can be easily proved using spectral sequences. By doing these exercises, you will lose any fear of spectral sequence arguments in similar situations, as you will realize they are all the same.

Before you read this section, you should read §1.7 on spectral sequences.

23.3.1. Symmetry of Tor**.**

23.3.A. EXERCISE (SYMMETRY OF Tor). Show that there is an isomorphism $\mathrm{Tor}_i^A(M, N) \overset{\sim}{\longleftrightarrow} \mathrm{Tor}_i^A(N, M)$. (Hint: take a free resolution of M and a free resolution of N. Take their "product" to somehow produce a double complex. Use both orientations of the obvious spectral sequence and see what you get.)

On a related note:

23.3.B. EXERCISE. Show that the two definitions of $\mathrm{Ext}^i(M, N)$ given in Exercises 23.2.D and 23.2.E agree.

23.3.2. Derived functors can be computed using acyclic resolutions. Suppose $F \colon \mathscr{A} \to \mathscr{B}$ is a right-exact additive functor of abelian categories, and that $\mathscr{A}$ has enough projectives. We say that $A \in \mathscr{A}$ is F**-acyclic** (or just **acyclic** if the F is clear from context) if $L_iF\, A = 0$ for $i > 0$. In Exercise 23.3.D, we will see that derived functors $L_iF\, B$ of an object B of $\mathscr{A}$ can be computed by using "acyclic resolutions". We set the stage with a useful construction.

23.3.3. *Building a "projective resolution" of an exact sequence.* Suppose $\cdots \to E_2 \to E_1 \to E_0 \to 0$ is an exact sequence in an abelian category with enough projectives.

We explain how to inductively build a double complex of projectives

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & P_{2,1} & \longrightarrow & P_{1,1} & \longrightarrow & P_{0,1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & P_{2,0} & \longrightarrow & P_{1,0} & \longrightarrow & P_{0,0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

such that the rows and columns are all exact. Suppose you have built part of the complex, and are trying to build the $P_{m,n}$ term:

(23.3.3.1)
$$\begin{array}{ccccc} ? & \longrightarrow & P_{m-1,n} & \longrightarrow & P_{m-2,n} \\ \downarrow & & \downarrow & & \downarrow \\ P_{m,n-1} & \longrightarrow & P_{m-1,n-1} & \longrightarrow & P_{m-2,n-1} \\ \downarrow & & \downarrow & & \\ P_{m,n-2} & \longrightarrow & P_{m-1,n-2} & & \end{array}$$

For reasons that will soon become clear, we assume that for any $b \in P_{m-1,n-1}$ whose image in both $P_{m-1,n-2}$ and $P_{m-2,n-1}$ is zero, there is $c \in P_{m,n-1}$ whose image in $P_{m-1,n-1}$ is b, and whose image in $P_{m,n-2}$ is zero; and symmetrically there is $c' \in P_{m-1,n}$ whose image in $P_{m-1,n-1}$ is b, and whose image in $P_{m-2,n}$ is zero.

Consider

$$K := \ker(P_{m,n-1} \oplus P_{m-1,n} \to P_{m,n-2} \oplus P_{m-1,n-1} \oplus P_{m-2,n}).$$

If K surjects onto both $\ker(P_{m-1,n} \to P_{m-2,n})$ and $\ker(P_{m,n-1} \to P_{m,n-2})$, then we could take any surjection from a projective object $P \twoheadrightarrow K$, then take $P_{m,n}$ to be P (with its map to $P_{m,n-1}$ and the negative of its map to $P_{m-1,n}$). With this choice, we would have ensured "horizontal exactness" at $P_{m-1,n}$, and "vertical exactness" at $P_{m,n-1}$, and commutativity of the square in (23.3.3.1).

We now verify our two desired surjections, by "diagram-chasing" (which we may do, see §1.6.5). Suppose $a \in \ker(P_{m-1,n} \to P_{m-2,n})$, and let b be its image in $P_{m-1,n-1}$. We wish to find $c \in \ker(P_{m,n-1} \to P_{m,n-2})$ so that c maps to b in $P_{m-1,n-1}$ and 0 in $P_{m,n-2}$. (Then $(-c, a) \in P_{m,n-1} \oplus P_{m-1,n}$ would be our desired element of K mapping to $a \in P_{m-1,n}$). But such a c exits by our assumption! And symmetrically, we get surjectivity $K \twoheadrightarrow \ker(P_{m,n-1} \to P_{m,n-2})$.

The final task is to ensure these assumptions hold for later stages in the building of our double complex. We need to ensure that for any $b' \in P_{m-1,n}$ that maps to $(0,0) \in P_{m-1,n-1} \oplus P_{m-2,n}$, there is an element of $P_{m,n}$ mapping to

$(0, b') \in P_{m,n-1} \oplus P_{m-1,n}$. And we have to ensure the analogous statement with the roles of m and n reversed. We take $Q' = \ker(P_{m-1,n} \to P_{m-1,n-1} \oplus P_{m-2,n})$, and $Q'' = \ker(P_{m,n-1} \to P_{m,n-2} \oplus P_{m-1,n-1})$, and take surjections $P' \twoheadrightarrow Q'$ and $P'' \twoheadrightarrow Q''$ from projective objects. Then take $P_{m,n}$ to be $P \oplus P' \oplus P''$, where the maps from P to $P_{m,n-1}$ and $P_{m-1,n}$ are as described above; the map P' to $P_{m-1,n}$ is the above-described map (via Q'); the map P' to $P_{m,n-1}$ is zero; and the opposite for P'' (the map to $P_{m-1,n}$ is zero, and the map to $P_{m,n-1}$ is as implied above, by way of Q''). The summand P' ensures our first desired assumption, and the summand P'' ensures our second.

23.3.C. EXERCISE. Verify that the above construction indeed gives a projective resolution of an exact sequence. Where did you use that the sequence $E_\bullet$ was exact?

Now let's apply this.

23.3.D. EXERCISE. Show that you can compute the derived functors of an objects B of $\mathscr{A}$ using **acyclic resolutions** (not just projective resolutions), i.e., by taking a resolution

$$\cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow B \longrightarrow 0 \qquad (23.3.3.2)$$

by F-acyclic objects A_i, truncating, applying F, and taking homology. Hence $\mathrm{Tor}_i(M, N)$ can be computed with a flat resolution of M or N. Hint: as describe above, build a double complex of projectives "on top of" the exact sequence (23.3.3.2). Remove the bottom row, and the right-most nonzero column, and then apply F, to obtain a new double complex. Use a spectral sequence argument to show that (i) the double complex has homology equal to $L_iF(B)$, and (ii) the homology of the double complex agrees with the construction given in the statement of the exercise. If this is too confusing, read more about the Cartan-Eilenberg resolution below.

23.3.4. The Grothendieck composition-of-functors spectral sequence.

Suppose $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are abelian categories, $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{B} \to \mathscr{C}$ are a left-exact additive covariant functors, and $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. Thus right derived functors of F, G, and $G \circ F$ exist. A reasonable question is: how are they related?

23.3.5. Theorem (Grothendieck composition-of-functors spectral sequence). — *Suppose $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{B} \to \mathscr{C}$ are left-exact additive covariant functors, and $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. Suppose further that F sends injective elements of $\mathscr{A}$ to G-acyclic elements of $\mathscr{B}$. Then for each $X \in \mathscr{A}$, there is a spectral sequence with ${}_{\rightarrow}E_2^{p,q} = R^qG(R^pF(X))$ converging to $R^\bullet(G \circ F)(X)$.*

We will soon see the Leray spectral sequence as an application (Theorem 23.4.5).

There is more one might want to extract from the proof of Theorem 23.3.5. For example, although E_0 page of the spectral sequence will depend on some choices (of injective resolutions), the E_2 page will be independent of choice. For our applications, we won't need this refinement.

We will have to work to establish Theorem 23.3.5, so the proof is possibly best skipped on a first reading.