($=$ char $K(X)$) is zero. Then there is a nonempty (=dense) open set $U \subset X$ such that $\pi|_U$ is smooth of relative dimension trdeg $K(X)/K(Y)$.
(b) Let k be a field of characteristic 0, and let $\pi\colon X \to Y$ be a dominant morphism of integral k-varieties. Then there is a nonempty (=dense) open set $U \subset X$ such that $\pi|_U$ is smooth of relative dimension $\dim X - \dim Y$.

21.6.C. EXERCISE (PROMISED IN EXERCISE 12.2.D). Suppose $f_1, \dots, f_n \in k[x_1, \dots, x_n]$, where k is a field of characteristic 0. Show that $f_1, \dots, f_n$ are algebraically dependent if and only if the determinant of the Jacobian matrix is identically zero. Hint: Why is this exercise given just after Theorem 21.6.4?

21.6.5. *Example.* Theorem 21.6.4 fails in positive characteristic: consider the purely inseparable extension $\mathbb{F}_p(t)/\mathbb{F}_p(t^p)$. The same problem can arise even over an *algebraically closed* field of characteristic p: consider $\mathbb{A}^1_k = \operatorname{Spec} k[t] \to \operatorname{Spec} k[u] = \mathbb{A}^1_k$, given by $u \mapsto t^p$.

Proof. Let $n = \operatorname{trdeg} K(Y)/K(X)$. We may replace Y by an affine open subset $\operatorname{Spec} B$, and then replace X by an affine open subset $\operatorname{Spec} A$, so π corresponds to the ring map $B \to A$. Choose n elements $x_1, \dots, x_n \in A$ that form a transcendence basis for $K(A)/K(B)$. (Do you see why we may choose them to lie in A?) Choose additional elements $x_{n+1}, \dots, x_N \in A$ so that $x_1, \dots, x_N$ generate A as a B-algebra.

For $i = 0, \dots, n$, let F_i be the subfield of $K(A)$ generated over $K(B)$ by $x_1, \dots, x_i$. For example, $F_0 = K(B)$, $F_N = K(A)$, and $F_i/K(B)$ is a purely transcendental field extension for $i \leq n$. For $i = n+1, \dots, N$, the minimal polynomial for $x_i \in K(A)$ over F_{i-1} (an element of $F_{i-1}[t]$) may be interpreted as $m_i(x_1, \dots, x_{i-1}, t)$ for some $m_i \in K(B)[y_1, \dots, y_{i-1}, t]$. By multiplying m_i by the products of the denominators of its coefficients, we obtain $M_i \in B[y_1, \dots, y_{i-1}, t]$ so that $M_i(x_1, \dots, x_{i-1}, t) \in F_{i-1}[t]$ is also a minimal polynomial for $x_i \in K(A)$ over F_{i-1}. Thus

$$K(A) = K(B)(x_1, \dots, x_n)[y_{n+1}, \dots, y_N]/I, \tag{21.6.5.1}$$

where

$$I = (M_{n+1}(x_1, \dots, x_n, y_{n+1}), M_{n+2}(x_1, \dots, x_n, y_{n+1}, y_{n+2}), \dots, M_N(x_1, \dots, x_n, y_{n+1}, \dots, y_N)).$$

Define $A' := B[y_1, \dots, y_N]/(M_{n+1}(y_1, \dots, y_{n+1}), \dots, M_N(y_1, \dots, y_N))$.

Let $X' := \operatorname{Spec} A'$. Let $Z_1, \dots, Z_s$ be the irreducible components of X'. (There are finitely many by Noetherianity of B: Exercise 3.6.T and Proposition 3.6.15.) Precisely one of them, say $Z = Z_1$, dominates $\operatorname{Spec} B$ (because $A' \otimes_B K(B) \cong K(A)$ by (21.6.5.1)). Let $U' \subset X'$ be the open subset $Z_1 \setminus Z_1 \cap (Z_2 \cup \cdots Z_s)$.

At the generic point of U', the Jacobian matrix of $M_{n+1}(x_{n+1}), \dots, M_N(x_N)$ with respect to $x_1, \dots, x_N$ has corank n. (Here we use the separability of the minimal polynomials M_i.) Thus there is an open subset $U'' \subset U'$ where the Jacobian has corank n, so $U'' \to \operatorname{Spec} B$ is smooth of relative dimension n. Furthermore U'' is irreducible. Now U'' and X are birational, so by Proposition 7.5.5 there is an open subset $U \subset X$ that is isomorphic to an open subset of U''. This is the open subset we seek.

21.6.D. EXERCISE. Prove Theorem 21.6.4 (b) (using part (a)). □

If furthermore X is smooth, the situation is even better.