and any scheme maps to $\operatorname{Spec}\mathbb{Z}$, this implies that $\Omega_{\mathbb{P}^1_B/B} \cong \mathscr{O}_{\mathbb{P}^1_B}(-2)$ for *any* base scheme B.

(Also, as suggested by §18.5.2, this shows that $\Omega_{\mathbb{P}^1/k}$ is the dualizing sheaf for $\mathbb{P}^1_k$; see also Example 18.5.4. But given that we haven't yet proved Serre duality, this isn't so meaningful.)

Side Remark: the fact that the degree of the tangent bundle is 2 is related to the "Hairy Ball Theorem" (the dimension 2 case of [**Hat**, Thm. 2.28]).

21.3.3. Hyperelliptic curves. Throughout this discussion of hyperelliptic curves, we suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$, so we may apply the discussion of §19.5. Consider a double cover $\pi\colon C \to \mathbb{P}^1_k$ by a regular projective curve C, branched over $2g+2$ distinct points. We will use the explicit coordinate description of hyperelliptic curves of (19.5.2.1). In particular, π is unbranched at 0. By Theorem 19.5.1, C has genus g.

21.3.B. EXERCISE: DIFFERENTIALS ON HYPERELLIPTIC CURVES. What is the degree of the invertible sheaf $\Omega_{C/k}$? (Hint: let x be a coordinate on one of the coordinate patches of $\mathbb{P}^1_k$. Consider $\pi^* dx$ on C, and count poles and zeros. Use the explicit coordinates of §19.5. You should find that $\pi^* dx$ has $2g+2$ zeros and 4 poles, counted with multiplicity, for a total of $2g-2$.) Doing this exercise will set you up well for the Riemann-Hurwitz formula, in §21.4.

21.3.C. EXERCISE. Show that $h^0(C, \Omega_{C/k}) = g$ (and hence that the geometric genus of C is g) as follows.
(a) Show that any regular differential ω on $\operatorname{Spec} k[x,y]/(y^2 - f(x))$ (i.e., an element of $\Omega_{(k[x,y]/(y^2-f(x)))/k}$) preserved by the involution $y \mapsto -y$ is pulled back from a differential on $\operatorname{Spec} k[x]$. (Hint: make sense of the statement "ω/dx is a rational function on $\operatorname{Spec} k[x,y]/(y^2 - f(x))$ preserved by the involution $y \mapsto -y$" and show that ω/dx is a rational function in x.)
(b) Use (a) to show that any differential $\omega \in H^0(C, \Omega_{C/k})$ preserved by the involution $i : y \mapsto -y$ must be pulled back from $\mathbb{P}^1$ by π, and hence must be zero. Show that every differential $\omega \in H^0(C, \Omega_{C/k})$ satisfies $i^*\omega = -\omega$.
(c) Show that $\frac{dx}{y}$ is a (regular) differential on $\operatorname{Spec} k[x,y]/(y^2 - f(x))$. Show that for $0 \le i < g$, $x^i(dx)/y$ extends to a global differential ω_i on C (i.e., with no poles).
(d) Show that the ω_i ($0 \le i < g$) are linearly independent differentials, i.e., linearly independent in the vector space $H^0(C, \Omega_{C/k})$. (Hint: Let $\{p, q\} = \pi^{-1}(0)$. Show that the valuation of ω_i at both p and q is i. If $\omega := \sum_{j=s}^{g-1} a_j \omega_j$ is a nontrivial linear combination of the ω_i, with $a_j \in k$, and $a_s \neq 0$, show that the valuation of ω at p is s, and hence $\omega \neq 0$.)
(e) Show that the ω_i span the vector space of differentials $H^0(C, \Omega_{C/k})$. (Hint: if $\omega \in H^0(C, \Omega_{C/k})$ use (d) to show that there are unique a_i such that $\omega' := \omega - \sum_{i=0}^{g-1} a_i \omega_i$ vanishes at p to order $\geq g$. By (b), ω' also vanishes at q to the same order. Use Exercise 21.3.B to show that ω' must be zero.)

Hence $\Omega_{C/k}$ is an invertible sheaf of degree $2g-2$ with g sections.

21.3.D. ⋆ EXERCISE (TOWARD SERRE DUALITY). (You may later see this as an example of Serre duality in action.)
(a) Show that $h^1(C, \Omega_{C/k}) = 1$. Interpret a generator of $H^1(C, \Omega_{C/k})$ as $x^{-1} dx$. (In