$\mathbb{A}^1$, $C' \to \mathbb{A}^1$, where x is the coordinate on $\mathbb{A}^1$. This induces a quadratic field extension K over $k(x)$. As $\operatorname{char} k \neq 2$, this extension is Galois. Let $\sigma\colon K \to K$ be the Galois involution. Let y be a nonzero element of K such that $\sigma(y) = -y$, so 1 and y form a basis for K over the field $k(x)$, and are eigenvectors of σ. Now $\sigma(y^2) = y^2$, so $y^2 \in k(x)$. We can replace y by an appropriate $k(x)$-multiple so that y^2 is a polynomial, with no repeated factors, and monic. (This is where we use the hypothesis that k is algebraically closed, to get leading coefficient 1.)

Thus $y^2 = x^N + a_{N-1}x^{N-1} + \cdots + a_0$, where the polynomial on the right (call it $f(x)$) has no repeated roots. The Jacobian criterion for regularity (in the guise of Exercise 13.2.F) implies that this curve C'_0 in $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$ is regular. Then C'_0 is normal and has the same function field as C'. Thus C'_0 and C' are both normalizations of $\mathbb{A}^1$ in the finite extension of fields generated by y, and hence are isomorphic. Thus we have identified C' in terms of an explicit equation.

The branch points correspond to those values of x for which there is exactly one value of y, i.e., the roots of $f(x)$. In particular, $N = r$, and

$$f(x) = (x - p_1) \cdots (x - p_r),$$

where the p_i are interpreted as elements of $\overline{k}$.

Having mastered the situation over $\mathbb{A}^1$, we return to the situation over $\mathbb{P}^1$. We will examine the branched cover over the affine open set $\mathbb{P}^1 \setminus \{0\} = \operatorname{Spec} k[u]$, where $u = 1/x$. The previous argument applied to $\operatorname{Spec} k[u]$ rather than $\operatorname{Spec} k[x]$ shows that any such double cover must be of the form

$$\begin{aligned} C'' &= \operatorname{Spec} k[Z,u]/\left(Z^2 - (u - 1/p_1)\cdots(u-1/p_r)\right) \\ &= \operatorname{Spec} k[Z,u]/\left(\left((-1)^r \prod p_i\right) Z^2 - u^r f(1/u)\right) \\ &\longrightarrow \operatorname{Spec} k[u] = \mathbb{A}^1. \end{aligned}$$

So if there is a double cover over all of $\mathbb{P}^1$, it must be obtained by gluing C'' to C', "over" the gluing of $\operatorname{Spec} k[x]$ to $\operatorname{Spec} k[u]$ to obtain $\mathbb{P}^1$.

Thus in $K(C)$, we must have

$$z^2 = u^r f(1/u) = f(x)/x^r = y^2/x^r$$

(where z is obtained from Z by multiplying by a square root of $(-1)^r \prod p_i$) from which $z^2 = y^2/x^r$.

If r is even, considering $K(C)$ as generated by y and x, there are two possible values of z: $z = \pm y/x^{r/2}$. After renaming z by $-z$ if necessary, there is a single way of gluing these two patches together (we choose the positive square root).

If r is odd, the result follows from Exercise 19.5.A below.

19.5.A. EXERCISE. Suppose $\operatorname{char} k \neq 2$. Show that x does not have a square root in the field $k(x)[y]/(y^2 - f(x))$, where f is a polynomial with nonzero roots $p_1, \ldots, p_r$. (Possible hint: why is $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$?) Explain how this proves Proposition 19.5.2 in the case where r is odd.

□