18.8 ⋆ Serre's characterizations of ampleness and affineness

Theorem 16.2.2 gave a number of characterizations of ampleness, in terms of projective geometry, global generation, and the Zariski topology. Here is another characterization, this time cohomological, under Noetherian hypotheses. Because (somewhat surprisingly) we won't use this result, this section is starred.

18.8.1. Theorem (Serre's cohomological criterion for ampleness). — *Suppose A is a Noetherian ring, X is a proper A-scheme, and $\mathscr{L}$ is an invertible sheaf on X. Then the following are equivalent.*

(a-c) *The invertible sheaf $\mathscr{L}$ is ample on X (over A).*

(e) *For all coherent sheaves $\mathscr{F}$ on X, there is an n_0 such that for $n \geq n_0$, $H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) = 0$ for all $i > 0$.*

The label (a-c) is intended to reflect the statement of Theorem 16.2.2. We avoid the label (d) because it appeared in Theorem 16.2.6. (Aside: the "properness" assumption cannot be removed, see Example 18.7.6.) Before getting to the proof, we motivate this result by giving some applications. (As a warm-up, you can give a second solution to Exercise 16.2.G in the Noetherian case, using the affineness of π to show that $H^i(X, \mathscr{F} \otimes (\pi^* \mathscr{L})^{\otimes m}) = H^i(Y, (\pi_* \mathscr{F}) \otimes \mathscr{L}^{\otimes m})$.)

18.8.A. EXERCISE. Suppose X is a proper A-scheme (A Noetherian), and $\mathscr{L}$ is an invertible sheaf on X. Show that $\mathscr{L}$ is ample on X if and only if $\mathscr{L}|_{X^{\text{red}}}$ is ample on X^{red}. Hint: for the "only if" direction, use Exercise 16.2.G. For the "if" direction, let $\mathscr{I}$ be the ideal sheaf cutting out the closed subscheme X^{red} in X. Filter $\mathscr{F}$ by powers of $\mathscr{I}$:

$$0 = \mathscr{I}^r \mathscr{F} \subset \mathscr{I}^{r-1} \mathscr{F} \subset \cdots \subset \mathscr{I} \mathscr{F} \subset \mathscr{F}.$$

(Essentially the same filtration appeared in Exercise 18.4.L, for similar reasons.) Show that each quotient $\mathscr{I}^n \mathscr{F} / \mathscr{I}^{n-1} \mathscr{F}$, twisted by a high enough power of $\mathscr{L}$, has no higher cohomology. Use descending induction on n to show each part $\mathscr{I}^n \mathscr{F}$ of the filtration (and hence in particular $\mathscr{F}$) has this property as well.

18.8.B. EXERCISE. Suppose X is a proper A-scheme (A Noetherian), and $\mathscr{L}$ is an invertible sheaf on X. Show that $\mathscr{L}$ is ample on X if and only if $\mathscr{L}$ is ample on each component. Hint: follow the outline of the solution to the previous exercise, taking instead $\mathscr{I}$ as the ideal sheaf of one component. Perhaps first reduce to the case where $X = X^{\text{red}}$.

18.8.C. TRICKY EXERCISE. Suppose C is a proper *reduced* curve (over a field k), and $\mathscr{L}$ is a line bundle on C, such that $\nu^* \mathscr{L}$ is an ample line bundle on the normalization $\tilde{C}$ (where the normalization as usual is $\nu : \tilde{C} \to C$). Show that $\mathscr{L}$ is ample on C, as follows. First read §17.1.6 to understand the morphism $\nu^{!?} : \mathit{QCoh}_C \to \mathit{QCoh}_{\tilde{C}}$. Then for every quasicoherent sheaf $\mathscr{F}$ on C, describe an exact sequence of quasicoherent sheaves on C:

$$0 \longrightarrow \mathscr{T} \longrightarrow \nu_* \nu^{!?} \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{Q} \longrightarrow 0$$

where $\mathscr{T}$ and $\mathscr{Q}$ are both supported on a set of dimension 0 (a finite set of points). Then show that for $n \gg 0$, $H^i(C, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) = 0$ for $i > 0$. You will use the projection formula (Exercise 18.7.E(b)) to show that $H^i(C, (\nu_* \nu^{!?} \mathscr{F}) \otimes \mathscr{L}^{\otimes n}) = 0$ for $i > 0$ and $n \gg 0$.