for a quasicoherent sheaf of algebras $\mathscr{S}_\bullet$ on Y (satisfying "finite generation in degree 1", Hypotheses 17.2.1). We say X is a **projective Y-scheme**, or X is **projective over** Y. Using Exercise 7.4.D, this generalizes the notion of a projective A-scheme.

17.3.2. *Warnings.* First, notice that $\mathscr{O}(1)$, an important part of the concept of $\mathscr{P}roj$, is not mentioned in the definition. (I would prefer that it be part of the definition, but this isn't accepted practice.) As a result, the notion of affine morphism is affine-local on the target, but the notion of projectivity of a morphism is not clearly affine-local on the target. (In Noetherian circumstances, with the additional data of the invertible sheaf $\mathscr{O}(1)$, it is, as we will see in §17.3.4. We will also later see an example showing that the property of being projective is *not* local on the target, §24.8.7.)

Second, [**Ha1**, p. 103] gives a different definition of projective morphism; we follow the more general definition of Grothendieck. These definitions turn out to be the same in nice circumstances. (But finite morphisms are not always projective in the sense of [**Ha1**], while they *are* projective in our sense.)

17.3.A. EXERCISE.
(a) *(a useful characterization of projective morphisms)* Suppose $\pi\colon X \to Y$ is a morphism. Show that π is projective if and only if there exist a finite type quasicoherent sheaf $\mathscr{S}_1$ on Y, and a closed embedding $i\colon X \hookrightarrow \mathscr{P}roj_Y \operatorname{Sym}^\bullet \mathscr{S}_1$ (over Y, i.e., commuting with the maps to Y). Hint: Exercise 17.2.H.
(b) *(a useful characterization of projective morphisms, with line bundle)* Suppose $\mathscr{L}$ is an invertible sheaf on X, and $\pi\colon X \to Y$ is a morphism. Show that π is projective, with $\mathscr{O}(1) \cong \mathscr{L}$, if and only if there exist a finite type quasicoherent sheaf $\mathscr{S}_1$ on Y, a closed embedding $i\colon X \hookrightarrow \mathscr{P}roj_Y \operatorname{Sym}^\bullet \mathscr{S}_1$ (over Y, i.e., commuting with the maps to Y), and an isomorphism $i^*\mathscr{O}_{\mathscr{P}roj_Y \operatorname{Sym}^\bullet \mathscr{S}_1}(1) \stackrel{\sim}{\longleftrightarrow} \mathscr{L}$.
(c) Suppose, furthermore, that Y admits an ample line bundle in the sense of §16.2.5, as is the case whenever Y is projective, affine or, more generally, quasiprojective. Show that π is projective if and only if there exists a closed embedding $X \to \mathbb{P}^n_Y$ (over Y) for some n. (If you want to avoid the starred section §16.2.5, you can assume that Y is projective over $\operatorname{Spec} A$ and use the definition of ample from §16.2.1. You will then have dealt with the important case where Y is projective, but missed out on other potentially interesting cases, such as when Y is affine or otherwise quasiprojective (but not proper).) Hint: the harder direction is the forward implication. Use the finite type quasicoherent sheaf $\mathscr{S}_1$ from (a). Tensor $\mathscr{S}_1$ with a high enough power of $\mathscr{M}$ so that it is finitely globally generated (Theorem 16.2.6, or Theorem 16.2.2 in the proper setting), to obtain a surjection

$$\mathscr{O}_Y^{\oplus(n+1)} \twoheadrightarrow \mathscr{S}_1 \otimes \mathscr{M}^{\otimes N}.$$

Then use Exercise 17.2.G.

17.3.3. *Definition: Quasiprojective morphisms.* In analogy with projective and quasiprojective A-schemes (§4.5.10), one may define quasiprojective morphisms. *If Y is quasicompact*, we say that $\pi\colon X \to Y$ is a **quasiprojective morphism** if π can be expressed as a quasicompact open embedding into a scheme projective over Y. This is not a great notion, and we will not use it. (The general definition of quasiprojective morphism is slightly delicate — see [**Gr-EGA**, II.5.3] — and we won't need it.)