17.2.F. EXERCISE. Suppose $\mathscr{S}_\bullet$ is finitely generated in degree 1 (Hypotheses 17.2.1). Describe a map of graded quasicoherent sheaves $\phi\colon \mathscr{S}_\bullet \to \oplus_{n=0}^{\infty} \beta_* \mathscr{O}(n)$ (β is the structure morphism, see Exercise 17.2.C). Hint: Exercise 15.6.C.

17.2.G. EXERCISE. Suppose $\mathscr{L}$ is an invertible sheaf on X, and $\mathscr{S}_\bullet$ is a quasicoherent sheaf of graded algebras on X generated in degree 1 (Hypotheses 17.2.1). Define $\mathscr{S}'_\bullet = \oplus_{n=0}^{\infty} (\mathscr{S}_n \otimes \mathscr{L}^{\otimes n})$. Then $\mathscr{S}'_\bullet$ has a natural algebra structure inherited from $\mathscr{S}_\bullet$; describe it. Give a natural isomorphism of "X-schemes with line bundles"

$$(\mathcal{P}roj\, \mathscr{S}'_\bullet, \mathscr{O}_{\mathcal{P}roj\, \mathscr{S}'_\bullet}(1)) \overset{\sim}{\longleftrightarrow} (\mathcal{P}roj\, \mathscr{S}_\bullet, \mathscr{O}_{\mathcal{P}roj\, \mathscr{S}_\bullet}(1) \otimes \beta^* \mathscr{L}),$$

where $\beta\colon \mathcal{P}roj\, \mathscr{S}_\bullet \to X$ is the structure morphism. In other words, informally speaking, the $\mathcal{P}roj$ is the same, but the $\mathscr{O}(1)$ is twisted by $\mathscr{L}$.

17.2.3. *Definition.* If $\mathscr{F}$ is a finite rank locally free sheaf on X, then $\mathcal{P}roj(\operatorname{Sym}^\bullet \mathscr{F}^\vee)$ is called its **projectivization**, and is denoted $\mathbb{P}\mathscr{F}$. (The reason for the dual is the same as for $\mathcal{S}pec(\operatorname{Sym}^\bullet \mathscr{F}^\vee)$ in Definition 17.1.5.) You can check that this construction behaves well with respect to base change. Define $\mathbb{P}^n_X := \mathbb{P}(\mathscr{O}_X^{\oplus(n+1)})$. (Then $\mathbb{P}^n_{\operatorname{Spec} A}$ agrees with our earlier definition of $\mathbb{P}^n_A$, cf. Exercise 4.5.Q, and $\mathbb{P}^n_X$ agrees with our earlier usage, see for example the proof of Theorem 11.5.5.) If $\mathscr{F}$ is locally free of rank $n+1$, then $\mathbb{P}\mathscr{F}$ is a **projective bundle** or $\mathbb{P}^n$**-bundle** over X. By Exercise 17.2.G, if $\mathscr{F}$ is a finite rank locally free sheaf on X, there is a canonical isomorphism $\mathbb{P}\mathscr{F} \overset{\sim}{\longleftrightarrow} \mathbb{P}(\mathscr{L} \otimes \mathscr{F})$.

More generally, if $\mathscr{F}$ is a finite type quasicoherent sheaf on X, then one might define similarly its projectivization $\mathcal{P}roj(\operatorname{Sym}^\bullet \mathscr{F}^\vee)$. Be careful, though. For example, if $\mathscr{G}$ is a torsion sheaf on an integral scheme, then $\mathscr{F}^\vee = 0$, so with this definition, $\mathbb{P}\mathscr{F} = \varnothing$. So this isn't a great notion.

Because there is not universal agreement on whether $\mathbb{P}\mathscr{F}$ should be defined as $\operatorname{Sym}^\bullet \mathscr{F}$ or $\operatorname{Sym}^\bullet \mathscr{F}^\vee$, parallel to whether there can be disagreement as to whether the projectivization of a vector space parametrizes one-dimensional subspaces or quotients (cf. Exercise 17.2.I), it is safest to avoid the notation $\mathbb{P}\mathscr{F}$, or at least to state at the outset which convention you are following.

17.2.4. *Example: ruled surfaces.* If C is a regular curve and $\mathscr{F}$ is locally free of rank 2, then $\mathbb{P}\mathscr{F}$ is called a **ruled surface** over C. If C is further isomorphic to $\mathbb{P}^1$, $\mathbb{P}\mathscr{F}$ is called a **Hirzebruch surface**. All vector bundles on $\mathbb{P}^1$ split as a direct sum of line bundles (see §18.5.5 for a proof), so each Hirzebruch surface is of the form $\mathbb{P}(\mathscr{O}(n_1) \oplus \mathscr{O}(n_2))$. By Exercise 17.2.G, this depends only on $n_2 - n_1$. The Hirzebruch surface $\mathbb{P}(\mathscr{O} \oplus \mathscr{O}(n))$ ($n \geq 0$) is often denoted $\mathbb{F}_n$. We will discuss the Hirzebruch surfaces in greater length in §20.2.10. We will see that the $\mathbb{F}_n$ are all distinct in Exercise 20.2.Q.

17.2.H. EXERCISE. If $\mathscr{S}_\bullet$ is finitely generated in degree 1 (Hypotheses 17.2.1), describe a canonical closed embedding

$$\begin{array}{ccc} \mathcal{P}roj\, \mathscr{S}_\bullet & \overset{i}{\hookrightarrow} & \mathcal{P}roj\, \operatorname{Sym}^\bullet \mathscr{S}_1 \\ & {\scriptstyle\beta}\searrow \quad \swarrow & \\ & X & \end{array}$$

and an isomorphism $\mathscr{O}_{\mathcal{P}roj\,\mathscr{S}_\bullet}(1) \xleftrightarrow{\sim} i^*\mathscr{O}_{\mathbb{P}\mathscr{S}_1}(1)$ arising from the surjection

$$\operatorname{Sym}^\bullet \mathscr{S}_1 \twoheadrightarrow \mathscr{S}_\bullet .$$

In particular, if $\mathscr{S}_1$ is locally free, then $\mathcal{P}roj \operatorname{Sym}^\bullet \mathscr{S}_1 = \mathbb{P}\mathscr{S}_1^\vee$, so we have embedded $\mathcal{P}roj\,\mathscr{S}_\bullet$ in a projective bundle on X.

17.2.I. EXERCISE. Suppose $\mathscr{F}$ is a locally free sheaf of rank $n+1$ on X. Exhibit a bijection between the set of sections $s\colon X \to \mathbb{P}\mathscr{F}$ of $\mathbb{P}\mathscr{F} \to X$ and the set of surjective homomorphisms $\mathscr{F} \to \mathscr{L} \to 0$ of $\mathscr{F}$ onto invertible sheaves on X. This functorial description of $\mathbb{P}\mathscr{F}$ in some sense generalizes the functorial description of projective space in §15.2.3.

17.2.5. *Remark (the relative version of the projective and affine cone).* There is a natural morphism from $\mathcal{S}pec\,\mathscr{S}_\bullet$ minus the zero-section to $\mathcal{P}roj\,\mathscr{S}_\bullet$ (cf. Exercise 9.3.N). Just as $\operatorname{Proj} S_\bullet[T]$ contains a closed subscheme identified with $\operatorname{Proj} S_\bullet$ whose complement can be identified with $\operatorname{Spec} S_\bullet$ (Exercise 9.3.O), $\mathcal{P}roj\,\mathscr{S}_\bullet[T]$ contains a closed subscheme identified with $\mathcal{P}roj\,\mathscr{S}_\bullet$ whose complement can be identified with $\mathcal{S}pec\,\mathscr{S}_\bullet$. You are welcome to think this through.

17.3 Projective morphisms

In §17.1, we reinterpreted affine morphisms: $X \to Y$ is an affine morphism if there is an isomorphism $X \xleftrightarrow{\sim} \mathcal{S}pec\,\mathscr{B}$ of Y-schemes for some quasicoherent sheaf of algebras $\mathscr{B}$ on Y. We will *define* the notion of a projective morphism similarly.

You might think that because projectivity is such a classical notion, there should be some obvious definition, that is reasonably behaved. But this is not the case, and there are many possible variant definitions of projective (see [**Stacks**, tag 01W8]). All are imperfect, including the accepted definition we give here. Although projective morphisms are preserved by base change, we will manage to show that they are preserved by composition only when the target is quasicompact (Exercise 17.3.B), and we will only show that the notion is local on the target when we add the data of a line bundle, and even then only under locally Noetherian hypotheses (§17.3.4).

17.3.1. *Definition.* A morphism $\pi\colon X \to Y$ is **projective** if there is an isomorphism

$$\begin{array}{ccc} X & \xrightarrow{\ \sim\ } & \mathcal{P}roj\,\mathscr{S}_\bullet \\ & {\scriptstyle \pi}\searrow \qquad \swarrow & \\ & Y & \end{array}$$

for a quasicoherent sheaf of algebras $\mathscr{S}_\bullet$ on Y (satisfying "finite generation in degree 1", Hypotheses 17.2.1). We say X is a **projective Y-scheme**, or X is **projective over** Y. Using Exercise 7.4.D, this generalizes the notion of a projective A-scheme.

17.3.2. *Warnings.* First, notice that $\mathscr{O}(1)$, an important part of the concept of $\mathcal{P}roj$, is not mentioned in the definition. (I would prefer that it be part of the definition,