**15.4.14.** By Exercise 15.4.G, $[Z] \mapsto \mathscr{O}(1)$, and as $\mathscr{O}(m)$ is nontrivial for $m \neq 0$ (Exercise 15.1.B), $[Z]$ is not torsion in $\operatorname{Cl} \mathbb{P}^n_k$. Hence $\operatorname{Pic}(\mathbb{P}^n_k) \hookrightarrow \operatorname{Cl}(\mathbb{P}^n_k)$ is an isomorphism, and $\boxed{\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}}$, with generator $\mathscr{O}(1)$. The **degree** of an invertible sheaf on $\mathbb{P}^n$ is defined using this: define $\deg \mathscr{O}(d)$ to be $d$. (You will have already proved that $\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ if you did Exercise 15.4.H; but we will use the strategy here to great effect in §15.5.4.)

We have gotten good mileage from the fact that the Picard group of the spectrum of a unique factorization domain is trivial. More generally, Exercise 15.4.K gives us:

**15.4.15. Proposition. —** *If $X$ is Noetherian and factorial, then for any Weil divisor $D$, $\mathscr{O}(D)$ is invertible, and hence the map $\operatorname{Pic} X \to \operatorname{Cl} X$ is an isomorphism.*

This can be used to make the connection to the class group in number theory precise, see Exercise 14.2.K; see also §15.5.5.

**15.4.16. Mild but important generalization: twisting line bundles by divisors.** The above constructions can be extended, with $\mathscr{O}_X$ replaced by an arbitrary invertible sheaf, as follows. Let $\mathscr{L}$ be an invertible sheaf on a normal Noetherian scheme $X$. Then define $\mathscr{L}(D)$ by $\mathscr{O}_X(D) \otimes \mathscr{L}$. If $D$ is locally principal, then $\mathscr{L}(D)$ is a line bundle. Notice that in this case there are two different ways of interpreting sections of $\mathscr{L}(D)$ over an open set, each with different advantages: as a section of the new line bundle $\mathscr{L}(D)$, and as rational sections of $\mathscr{L}$ with constraints on poles and zeros given by the divisor $D$.

**15.4.L.** EASY EXERCISE.
(a) Assume for convenience that $X$ is irreducible. Show that sections of $\mathscr{L}(D)$ can be interpreted as rational sections of $\mathscr{L}$ with zeros and poles constrained by $D$, just as in (15.4.5.1):

$$\Gamma(U, \mathscr{L}(D)) := \{t \text{ nonzero rational section of } \mathscr{L} \ : \ \operatorname{div}|_U t + D|_U \geq 0\} \cup \{0\}.$$

(b) Suppose $D_1$ and $D_2$ are locally principal. Show that

$$(\mathscr{O}(D_1))(D_2) \cong \mathscr{O}(D_1 + D_2).$$

**15.4.17. A variation of the Qcqs Lemma.** The Qcqs Lemma 6.2.9, proved in Exercise 6.2.G, has the following generalization.

**15.4.M.** IMPORTANT EXERCISE (TO BE USED REPEATEDLY). Suppose $X$ is a quasicompact quasiseparated scheme, $\mathscr{L}$ is an invertible sheaf on $X$ with section $s$, and $\mathscr{F}$ is a quasicoherent sheaf on $X$. Generalizing Definition 6.2.8, let $X_s$ be the open subset of $X$ where $s$ doesn't vanish. We interpret $s$ as a degree 1 element of the graded ring $R(\mathscr{L})_\bullet := \oplus_{n \geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$. Note that $\oplus_{n\geq 0}\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n})$ is a graded $R(\mathscr{L})_\bullet$-module.
(a) Describe a natural map

$$\left((\oplus_{n\geq 0}\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}))_s\right)_0 \longrightarrow \Gamma(X_s, \mathscr{F}).$$

(Possible hint: for quasicoherent sheaves, "tensor product has no need to be sheafified when restricted to affine subschemes", Exercise 6.2.F.)

(b) Show that this map is an isomorphism. (Hint: show this map is an isomorphism in the affine case.)

Translation: Any section of $\mathscr{F}$ over $X_s$ can be extended to a section over $X$ after multiplying by some some appropriate power of $s$. And if we have two such extensions, they become equal after multiplying by another appropriate power of $s$.

## 15.5 The pay-off: Many fun examples

**15.5.1. Fun examples: Projective transformations.**
The fact that $\operatorname{Pic} \mathbb{P}^n_k \cong \mathbb{Z}$ has many wonderful and cheap consequences.

**15.5.A.** EXERCISE (AUTOMORPHISMS OF PROJECTIVE SPACE). Show that all the automorphisms of projective space $\mathbb{P}^n_k$ (fixing $k$) correspond to $(n+1) \times (n+1)$ invertible matrices over $k$, modulo scalars (also known as $\mathrm{PGL}_{n+1}(k)$). (Hint: Suppose $\pi \colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that this induces an isomorphism $\pi^* \mathscr{O}(1) \xrightarrow{\sim} \mathscr{O}(1)$. Show that $\pi^* \colon \Gamma(\mathbb{P}^n, \mathscr{O}(1)) \to \Gamma(\mathbb{P}^n, \mathscr{O}(1))$ is an isomorphism.)

**15.5.2.** Automorphisms of projective space are often called **projective transformations**. Because of Exercise 15.5.A, in their incarnation of matrices modulo scalars, projective transformations are also called **projective changes of coordinates**. Exercise 15.5.A will be useful later, especially for the case $n = 1$. In this case, these automorphisms are called **fractional linear transformations**. (For experts: why was Exercise 15.5.A not stated over an arbitrary base ring $A$? Where does the argument go wrong in that case? For what rings $A$ does the result still work?)

**15.5.B.** EXERCISE. Show that $\operatorname{Aut}(\mathbb{P}^1_k)$ is strictly three-transitive on $k$-valued points, i.e., given two triplets $(p_1, p_2, p_3)$ and $(q_1, q_2, q_3)$ each of distinct $k$-valued points of $\mathbb{P}^1$, there is precisely one automorphism of $\mathbb{P}^1$ sending $p_i$ to $q_i$ ($i = 1, 2, 3$).

**15.5.C.** EXERCISE. Solve these problems over an arbitrary field $k$.
(a) Find a linear fractional transformation $f(t) \in \mathrm{PGL}(2)$ that has order precisely 3 in $\mathrm{PGL}(2)$.
(b) Show that any two order 3 elements of $\mathrm{PGL}(2)$ are conjugate. (Possible hint: use transitivity.)

**15.5.D.** EXERCISE. Suppose $p_0, \ldots, p_{n+1}$ are $n+2$ distinct $k$-valued points of $\mathbb{P}^n_k$, no $n+1$ of which lie on a hyperplane. Show that there is a unique projective transformation taking $p_i$ ($0 \leq i \leq n$) to $[0, \ldots, 0, 1, 0, \ldots, 0]$ (where the 1 is in the $i$th position), and taking $p_{n+1}$ to $[1, \ldots, 1]$.

**15.5.E.** FUN EXERCISE. Suppose $X$ is a quasiprojective $k$-scheme, and $\pi \colon \mathbb{P}^n_k \to X$ is any morphism (over $k$). Show that either the image of $\pi$ has dimension $n$, or $\pi$ contracts $\mathbb{P}^n_k$ to a point. In particular, there are no nonconstant maps from projective space to a smaller-dimensional quasi-projective variety. Hint: show that it suffices to assume $k$ is algebraically closed, and in particular, infinite. If $X \subset \mathbb{P}^N$,