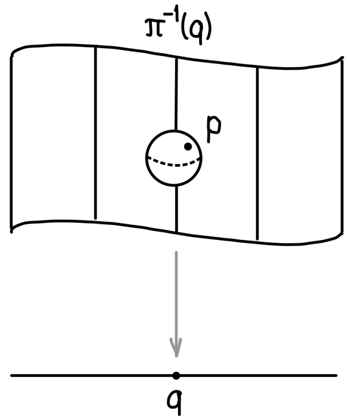


FIGURE 12.4. Exercise 12.4.A: the codimension of a point in the total space is bounded by the sum of the codimension of the point in the fiber plus the codimension of the image in the target

fails. We will see that equality always holds for sufficiently nice — flat — morphisms, see Proposition 24.6.6.

We now show that the inequality of Exercise 12.4.A is actually an equality over "most of $Y$".

**12.4.1. Theorem. —** *Suppose $\pi\colon X \to Y$ is a finite type dominant morphism of integral schemes such that the finitely generated field extension $K(X)/K(Y)$ has transcendence degree $r$. Then there exists a nonempty open subset $U \subset Y$ such that for all $q \in U$, the fiber over $q$ has pure dimension $r$.*

(By convention, the empty set technically has pure dimension $r$ — every irreducible component of the empty set has dimension $r$. This makes the statement of Theorem 12.4.1 a bit cleaner.) An interesting fact: we will use the Noether Normalization Lemma 12.2.4 in the proof, even though the statement doesn't involve varieties! In the generality of the statement bothers you, we have the following special case for varieties.

**12.4.2. Corollary. —** *Suppose $\pi\colon X \to Y$ is a (necessarily finite type) morphism of irreducible $k$-varieties, with $\dim X = m$ and $\dim Y = n$. Then there exists a nonempty open subset $U \subset Y$ such that for all $q \in U$, the fiber over $q$ has pure dimension $m - n$, or is empty.*

**12.4.B.** EXERCISE. Verify that Theorem 12.4.1 implies Corollary 12.4.2.

*Proof.* We begin with three quick reductions. (i) By shrinking $Y$ if necessary, we may assume that $Y$ is affine, say $\operatorname{Spec} B$. (ii) We may also assume that $X$ is affine, say $\operatorname{Spec} A$. (Reason: cover $X$ with a finite number of affine open subsets $X_1, \dots, X_a$, and take the intersection of the $U$'s for each of the $\pi|_{X_i}$.) (iii) If $\pi$ is not dominant, then we are done, as the image misses a dense open subset $U$ of $\operatorname{Spec} B$. So we assume now that $\pi$ is dominant.

In order to motivate the rest of the argument, we describe our goal. We will produce a nonempty distinguished open subset $U$ of $\operatorname{Spec} B$ so that $\pi^{-1}(U) \to U$ factors through $\mathbb{A}^r_U$ via a finite surjective morphism:

(12.4.2.1)
$$\begin{tikzcd}
X = & \operatorname{Spec} A \arrow[dd, "\pi"'] & \pi^{-1}(U) \arrow[l, hook', "\text{open emb.}"'] \arrow[d, "\text{finite surj.}"] \\
& & \mathbb{A}^r_U \arrow[d] \\
Y = & \operatorname{Spec} B & U \arrow[l, hook', "\text{open emb.}"']
\end{tikzcd}$$

**12.4.C.** EXERCISE. Show that this suffices to prove Theorem 12.4.1.

So we now work to build (12.4.2.1). We begin by noting that we have inclusions of $B$ into both $A$ and $K(B)$, and from both $A$ and $K(B)$ into $K(A)$. The maps from $A$ and $K(B)$ into $K(A)$ both factor through $A \otimes_B K(B)$ (whose Spec is the generic fiber of $\pi$), so the maps from both $A$ and $K(B)$ to $A \otimes_B K(B)$ must be inclusions.

(12.4.2.2)
$$\begin{tikzcd}
& & K(A) \\
A \arrow[r, "\therefore\hookrightarrow"'] \arrow[urr, hook] & A \otimes_B K(B) \arrow[ur] & \\
B \arrow[u, hook] \arrow[r, hook] & K(B) \arrow[u, "\therefore\hookrightarrow"] \arrow[uur, hook] &
\end{tikzcd}$$

Clearly $K(A) \otimes_B K(B) = K(A)$ (as $A \otimes_B K(B)$ can be interpreted as taking $A$ and inverting those nonzero elements of $B$), and $A \otimes_B K(B)$ is a finitely generated algebra over the field $K(B)$.

By Noether normalization 12.2.4, we can find elements $t_1, \dots, t_r \in A \otimes_B K(B)$, algebraically independent over $K(B)$, such that $A \otimes_B K(B)$ is integral over $K(B)[t_1, \dots, t_r]$.

Now, we can think of the elements $t_i \in A \otimes_B K(B)$ as fractions, with numerators in $A$ and (nonzero) denominators in $B$. If $f$ is the product of the denominators appearing for each $t_i$, then by replacing $B$ by $B_f$ (replacing $\operatorname{Spec} B$ by its distinguished open subset $D(f)$), we may assume that the $t_i$ are all in $A$. Thus (after sloppily renaming $B_f$ as $B$, and $A_f$ as $A$) we can trim and extend (12.4.2.2) to the following.

$$\begin{tikzcd}
A \arrow[r, hook] & A \otimes_B K(B) \\
B[t_1, \dots, t_r] \arrow[u, hook] \arrow[r, hook] & K(B)[t_1, \dots, t_r] \arrow[u, hook, "\text{integral}"'] \\
B \arrow[u, hook] \arrow[r, hook] & K(B) \arrow[u, hook]
\end{tikzcd}$$

Now $A$ is finitely generated over $B$, and hence over $B[t_1, \dots, t_r]$. But we cannot yet be sure that $A$ is *finite* over $B[t_1, \dots, t_r]$, so we are not done. We will have to localize $B$ further.

Suppose $A$ is generated over $B$ by $u_1, \dots, u_q$. Our Noether normalization argument implies that each $u_i$ satisfies some monic equation $f_i(u_i) = 0$, where $f_i \in K(B)[t_1, \dots, t_r][t]$. The coefficients of $f_i$ (considered as polynomials in $t_1, \dots, t_r$, $t$) are a priori fractions in $B$, but by multiplying by all those denominators, we can assume each $f_i \in B[t_1, \dots, t_r][t]$, at the cost of losing monicity of the $f_i$ (this time considered as polynomials in $t$). Let $b \in B$ be the product of the leading coefficients (considered as polynomials in $t$) of all the $f_i$. If $U = D(b)$ (the locus where $b$ is invertible), then over $U$, the $f_i$ (can be taken to) have leading coefficient $1$, so the $u_i$ (in $A_b$) are integral over $B_b[t_1, \dots, t_r]$. Thus $\operatorname{Spec} A_b \to \operatorname{Spec} B_b[t_1, \dots, t_r]$ is finite and surjective (the latter by the Lying Over Theorem 8.2.5).

We have now constructed (12.4.2.1), as desired. $\square$

There are a couple of things worth pointing out about the proof. First, although the result was originally motivated by classical varieties over a field $k$, the proof uses the theory of varieties over *another* field, the function field $K(B)$. This is an example of how the introduction of generic points to algebraic geometry is useful even for considering more "classical" questions.

Second, the idea of the main part of the argument is that we have a result over the generic point ($\operatorname{Spec} A \otimes_B K(B)$ is finite and surjective over affine space over $K(B)$), and we want to "spread it out" to an open neighborhood of the generic point of $\operatorname{Spec} B$. We do this by realizing that "finitely many denominators" appear when correctly describing the problem, and inverting those functions. This "spreading out" idea is a recurring theme.

**12.4.D.** EXERCISE (USEFUL CRITERION FOR IRREDUCIBILITY). Suppose $\pi: X \to Y$ is a proper morphism to an irreducible variety, and all the fibers of $\pi$ are nonempty, and irreducible of the same dimension. Show that $X$ is irreducible.

**12.4.3. Theorem (upper semicontinuity of fiber dimension). —** *Suppose $\pi: X \to Y$ is a morphism of finite type $k$-schemes.*
*(a) (upper semicontinuity on the source) The dimension of the fiber of $\pi$ at $p \in X$ (the dimension of the largest component of $\pi^{-1}(\pi(p))$ containing $p$) is an upper semicontinuous function in $p$ (i.e., on $X$).*
*(b) (upper semicontinuity on the target) If furthermore $\pi$ is closed (e.g., if $\pi$ is proper), then the dimension of the fiber of $\pi$ over $q \in Y$ is an upper semicontinuous function in $q$ (i.e., on $Y$).*

You should be able to immediately construct a counterexample to part (b) if the closedness hypothesis is dropped. (We also remark that Theorem 12.4.3(b) for projective morphisms is done, in a simple way, in Exercise 18.1.B.)

*Proof.* (a) Let $F_n$ be the subset of $X$ consisting of points where the fiber dimension is at least $n$. We wish to show that $F_n$ is a closed subset for all $n$. We argue by induction on $\dim Y$. The base case $\dim Y = 0$ is trivial. So we fix $Y$, and assume the result for all smaller-dimensional targets.

**12.4.E.** EXERCISE. Show that it suffices to prove the result when $X$ and $Y$ are integral, and $\pi$ is dominant.