(i) associativity axiom:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times id} & X \times X \\ {\scriptstyle id \times m}\downarrow & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow{m} & X \end{array}$$

commutes. (Here id means the equality $X \to X$.)

(ii) identity axiom:

$$X \xrightarrow{\sim} Z \times X \xrightarrow{e \times id} X \times X \xrightarrow{m} X$$

and

$$X \xrightarrow{\sim} X \times Z \xrightarrow{id \times e} X \times X \xrightarrow{m} X$$

are both the identity map $X \to X$. (This corresponds to the group axiom: "multiplication by the identity element is the identity map".)

(iii) inverse axiom: $X \xrightarrow{i \times id} X \times X \xrightarrow{m} X$ and $X \xrightarrow{id \times i} X \times X \xrightarrow{m} X$ are both the map that is the composition $X \longrightarrow Z \xrightarrow{e} X$.

As motivation, you can check that a group object in the category of sets is in fact the same thing as a group. (This is symptomatic of how you take some notion and make it categorical. You write down its axioms in a categorical way, and if all goes well, if you specialize to the category of sets, you get your original notion. You can apply this to the notion of "rings" in an exercise below.)

A **group scheme** is defined to be a group object in the category of schemes. A **group scheme** over a ring A (or a scheme S) is defined to be a group object in the category of A-schemes (or S-schemes).

7.6.G. EXERCISE. Give $\mathbb{A}^1_{\mathbb{Z}}$ the structure of a group scheme, by describing the three structural morphisms, and showing that they satisfy the axioms. (Hint: the morphisms should not be surprising. For example, inverse is given by $t \mapsto -t$. Note that we know that the product $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ exists, by Exercise 7.6.E(a).)

7.6.H. EXERCISE. Show that if G is a group object in a category $\mathscr{C}$, then for any $X \in \mathscr{C}$, $\mathrm{Mor}(X, G)$ has the structure of a group, and the group structure is preserved by pullback (i.e., $\mathrm{Mor}(\cdot, G)$ is a contravariant functor to the category of groups, *Groups*).

7.6.I. EXERCISE. Show that the group structure described by the previous exercise translates the group scheme structure on $\mathbb{A}^1_{\mathbb{Z}}$ to the group structure on $\Gamma(X, \mathscr{O}_X)$, via the bijection of §7.6.1.

7.6.J. EXERCISE. Define the notion of **abelian group scheme**, and **ring scheme**. (You will undoubtedly at the same time figure out how to define the notion of abelian group object and ring object in any category $\mathscr{C}$. You may discover a more efficient approach to such questions after reading §7.6.5.)

7.6.5. *Group schemes, more functorially.* There was something unsatisfactory about our discussion of the "group-respecting" nature of the bijection in §7.6.1: we observed that the right side (functions on X) formed a group, then we developed the axioms of a group scheme, then we cleverly figured out the maps that made