Compositions of separated morphisms are separated (Proposition 11.3.3), so we are done. $\square$

Here is an exercise we won't use, but you may like. It includes a converse to Proposition 11.3.13.

11.3.N. EXERCISE. Show that $X \to \operatorname{Spec} A$ is separated if and only if, for all affine open subsets U and V of X, (i) the intersection $U \cap V$ is affine, and (ii) the map $\mathscr{O}(U) \otimes_A \mathscr{O}(V) \to \mathscr{O}(U \cap V)$ is surjective. Show that it is enough to check that this holds as U and V range over the sets in any affine cover $X = \cup U_i$. (Hint: we largely did this in Proof 1 of Proposition 11.3.8.)

11.3.16. $\star\star$ **Universally injective morphisms and the diagonal.**

11.3.O. EXERCISE. Show that $\pi\colon X \to Y$ is universally injective if and only if the diagonal morphism $\delta_\pi\colon X \to X \times_Y X$ is surjective. (Recall that δ_π is *always* injective, by Proposition 11.3.1(b).)

Because surjective morphisms form a "reasonable" class (Exercise 10.4.G), we see that universally injective morphisms also form a "reasonable" class.

11.3.P. EASY EXERCISE. If $\pi\colon X \to Y$ and $\rho\colon Y \to Z$ are morphisms, and $\rho \circ \pi$ is universally injective, show that π is universally injective.

11.3.Q. EXERCISE.
(a) Show that universally injective morphisms are separated.
(b) Show that a map between finite type schemes over an algebraically closed field $\overline{k}$ is universally injective if and only if it is injective on closed points.

11.4 The locus where two morphisms from X to Y agree, and the "Reduced-to-Separated" Theorem

When we introduced rational maps in §7.5, we promised that in good circumstances, a rational map has a "largest domain of definition". We are now ready to make precise what "good circumstances" means, in the Reduced-to-Separated Theorem 11.4.2. We first introduce an important result making sense of locus where two morphisms with the same source and target "agree".

11.4.A. USEFUL EXERCISE: THE LOCUS WHERE TWO MORPHISMS AGREE. Suppose $\pi\colon X \to Y$ and $\pi'\colon X \to Y$ are two morphisms over some scheme Z.

$$\begin{tikzcd} X \arrow[rr, bend left, "\pi"] \arrow[rr, bend right, "\pi'"'] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

We can now give meaning to the phrase 'the locus where π and π' agree', and that in particular there is a largest locally closed subscheme where they agree — which is closed if Y is separated over Z. Suppose $\mu\colon W \to X$ is some morphism (not assumed to be a locally closed embedding). We say that π and π' agree on μ if